Simple Linear regression
Checking the model

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The assumptions.

Independent errors?

Predicted values and residuals.

Do the errors have the same distribution?

Normal errors?

Two examples, where the model is not valid.

Leverage: a measure of influence.

Standardized residuals.

Simple linear regression: The model

Let Y_i and x_i be the data for the ith person.

$$Y_i = \beta_0 + \beta_1 \cdot x_i + E_i \quad E_i \sim N\left(0, \sigma^2\right)$$

This model is based on the **assumptions**:

1. The **expected** value of Y is a **linear function** of x.
2. The **unexplained** random deviations are **independent**.
3. The unexplained random deviations have the **same distributions**.
4. This distribution is **normal**.

Checking the model: Independent errors ?

Assumption no. 2: *the errors should be **independent***, is mainly checked by considering **how the data was collected**.

The assumption is **violated** if

•some of the persons are **relatives** (and some are not) and the dependent variable have some **genetic** component.

•some of the persons were **measured** using one instrument and others using another.

•in general if the persons were sampled in **clusters**.

Predicted values and residuals

$$Y_i = \beta_0 + \beta_1 \cdot x_i + E_i \quad E_i \sim N\left(0, \sigma^2\right)$$

Based on the estimates we can calculate the **predicted** (fitted) values and the **residuals**:

$$\text{Predicted value:} \quad \hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 \cdot x_i$$

$$\text{Residual:} \quad r_i = y_i - \hat{y}_i = y_i - \left(\hat{\beta}_0 + \hat{\beta}_1 \cdot x_i\right)$$

The **predicted value** is the best guess of y_i (based on the estimates) for the ith person.

The **residual** is a guess of E_i (based on the estimates) for the ith person.

Stata:
```
predict PEFR_hat if e(sample),xb
predict PEFR_res if e(sample),resid
```

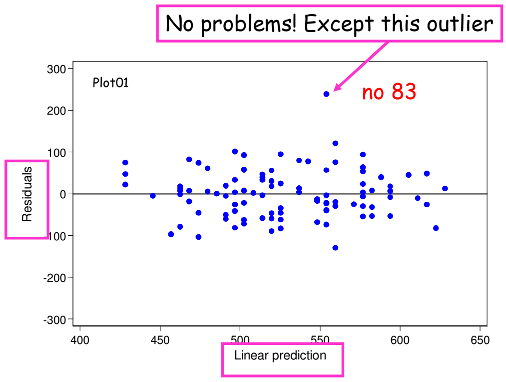

Checking the model:
Linearity and identical distributed errors

Assumption no. 1:
The **expected** value of Y is a **linear function** of x.
Assumption no. 3:
The unexplained random deviations have the **same distributions**.

These are checked by inspecting the following plots of:

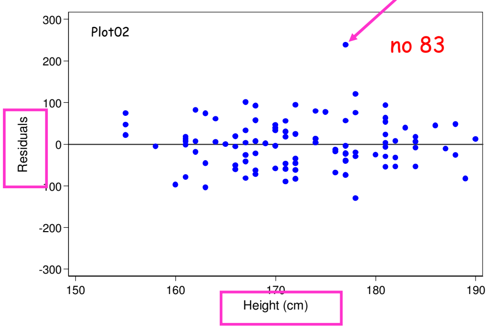
- **Residuals** versus **predicted**
- **Residuals** versus x

Checking the model:
Linearity and identical distributed errors



Checking the model: Linearity and identical distributed errors

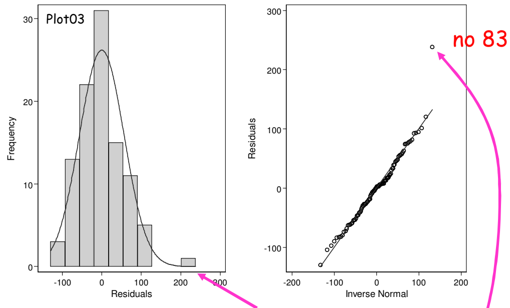
No problems! Except this outlier



Checking the model: Normal errors ?

Assumption no. 4: *the errors should be **normal distibuted**.*

This is checked by making histograms or qq-plots of the residuals:

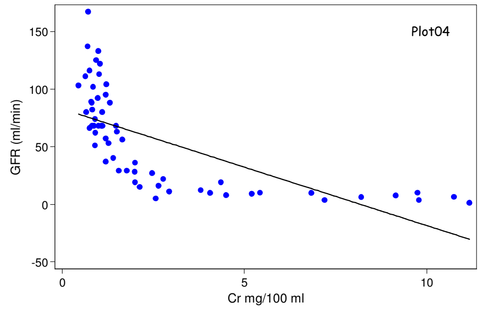


Not perfect, a bit skew.

Notice this point

Assumptions violated: Example 2

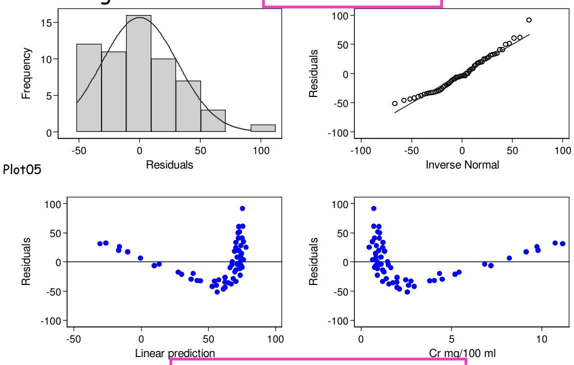
The relation between GFR and Serum Creatinine



Clearly non-linear!

Assumptions violated: Example 2

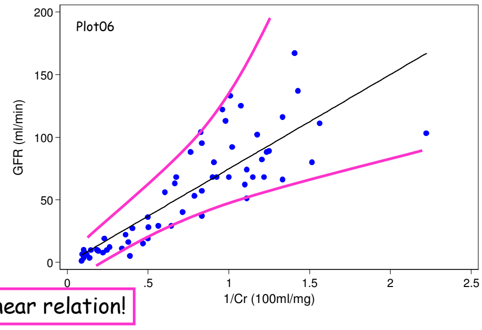
Checking the model **Close to normal**



Clearly not constant mean!

Assumptions violated: Example 3

The relation between GFR and 1/Serum Creatinine

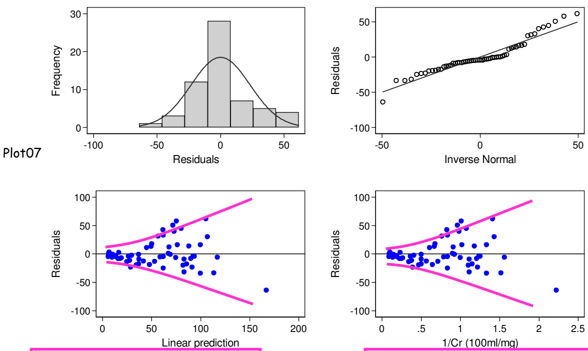


A linear relation!

Increasing variation!

Assumptions violated: Example 3

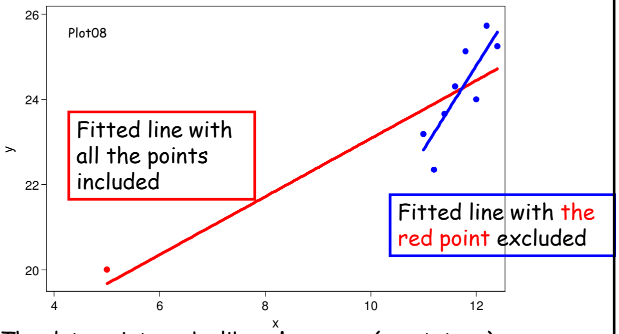
Checking the model **Close to normal**



Increasing variation! **Increasing variation!**

Influential data points: Example 4

Not all data points have the same influence on the estimates:



The data point works like a **leverage** (vægtstang).

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Influential data points: Leverage

The influence of a data point is sometime measured by its **leverage**:

$$h_i = \frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum_{j=1}^{n} (x_j - \bar{x})^2}$$

A large value implies that the estimates and/or the standard errors is **highly influenced** by this observation.

Note that $0 \le h_i \le 1$

Notice, it is a function only of the **independent** variable, x and the sample size.

The leverage for a given data point depends on **how far away** its **independent** variable is from the **average value**.

Stata: *predict* PEFR_lev *if e(sample), leverage*

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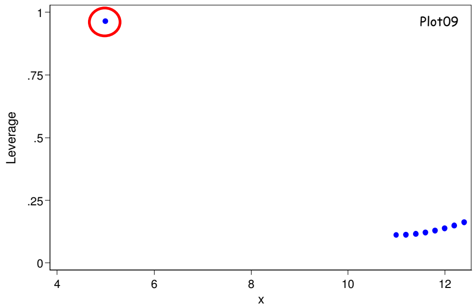
Influential data points Leverage

A **leverage** versus **independent variable** for the example on page 13.



The data point with the 'extreme' x value has very high leverage – as expected.

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Types of residuals: Standardized residuals

The (**unstandardized**) residual: $r_i = y_i - \left(\hat{\beta}_0 + \hat{\beta}_1 \cdot x_i\right)$

has mean zero but **non-constant** variance: $\text{sd}(r_i) = \sigma\sqrt{1-h_i}$

I. e. residuals from points with **high leverage** have **smaller variance**, than residuals from points with small leverage.

Due to this one often use the **standardized** residual:

$$z_i = \frac{r_i}{\hat{\sigma}\sqrt{1-h_i}}$$

This will have **variance 1**, if the **model is true**.

Stata: *predict* PEFR_zres *if e(sample),rstandard*

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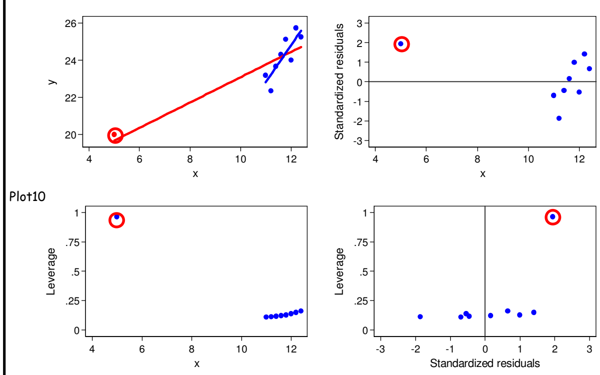
Influential data points? Example 4



Large leverage and standardized residual large!

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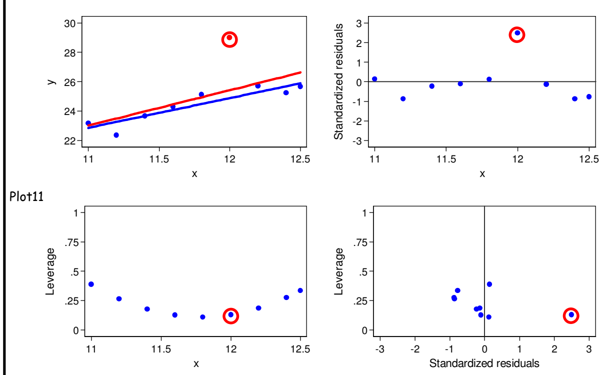
Influential data points? Example 5



Small leverage but standardized residual large!

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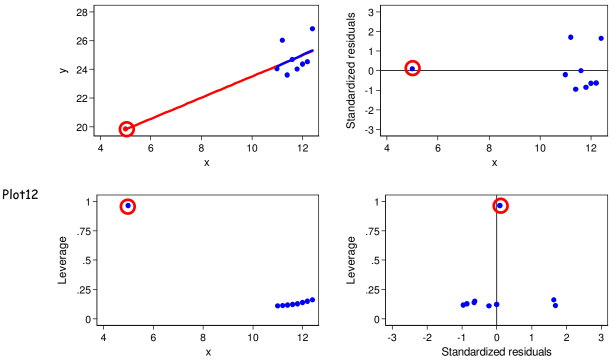
Influential data points? Example 6



Large leverage but standardized residual ok!

Influential data points? Example 6

Results with using all data:

```
Root MSE      =  1.0282
------------------------------------------------------------------------------
           y |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |   .7364484   .1594519     4.62   0.002     .3594045    1.113492
       _cons |    16.1386    1.78019     9.07   0.000     11.92912    20.34808
------------------------------------------------------------------------------
```

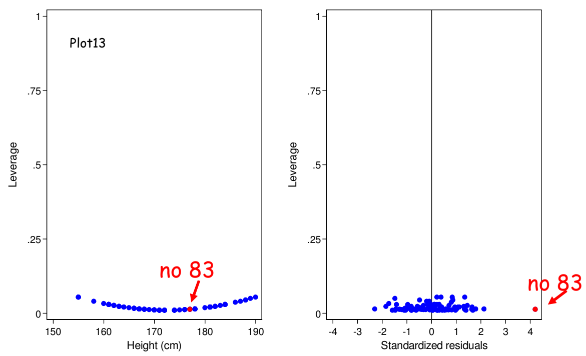
Results without the point with high leverage:

```
Root MSE      =  1.1099
------------------------------------------------------------------------------
           y |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |   .8080605   .8563254     0.94   0.382    -1.287292    2.903413
       _cons |    15.2985   10.02669     1.53   0.178    -9.235928    39.83292
------------------------------------------------------------------------------
```

Point estimates unchanged

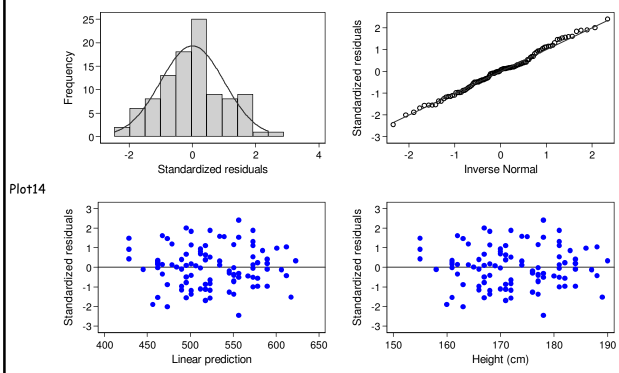
Standard errors much larger. **Confidence intervals** much wider.

The PEFR example: leverage and standardized residuals



Leverages are small, observation no. 83 has large residual

The PEFR example: Excluding observation no 83



Some comments on checking a (simple) linear regression

Always consider the design: **How was the data collected**?

This has implications for the validity of the **statistical model**.

And it has implications for the **interpretation** of the results.

Observations with **high leverages** have 'extreme' values of the **independent** variable.

These observation will have **high impact** on the results, but might not be 'representative'.

Sometimes it is best to **exclude** these from the analysis.

Observation with **large residuals**, that is observed y value far away from expected, should be **checked for errors**.

Prediction interval for future value

The **true line** is given as : $y = \beta_0 + \beta_1 \cdot x$

and **estimated** by plugging in the estimates $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 \cdot x$

The standard **deviation** for a **new observation** is given by:

$$\mathrm{sd}\left(\hat{\beta}_0 + \hat{\beta}_1 \cdot x + E\right) = \hat{\sigma}\sqrt{1 + \frac{1}{n} + \frac{(x-\bar{x})^2}{\sum (x_i - \bar{x})^2}}$$

with the 95% (pointwise) **prediction interval**

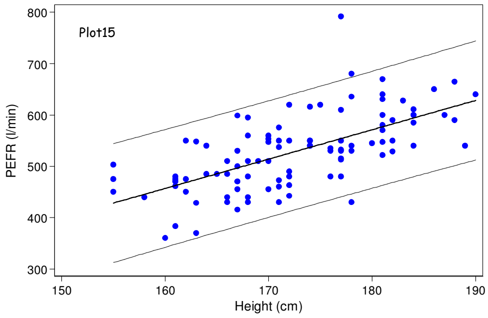
$$\hat{\beta}_0 + \hat{\beta}_1 \cdot x \pm t_{n-2}^{0.975} \cdot \mathrm{sd}\left(\hat{\beta}_0 + \hat{\beta}_1 \cdot x + E\right)$$

Many programs can make a plot with the fitted line and its prediction limits.

In Stata its done by the ***lfitci*** and graph command, the option ***stdf***

Prediction interval for future value

```
twoway ///
 (scatter PEFR height, mco(blue) msym(O))          ///
 (lfitci PEFR height, stdf clpat(1) cip(rline) )   ///
  ,legend(off) ytit("PEFR (l/min)")
```



Morten Frydenberg Linear and Logistic regression - Note 1.2 25