Multiple linear regression 1

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Why do we need multiple linear regression.

An example
- Interpretation of the parameters

The general model
- The assumptions.
- The parameters.
- Estimation.
- The distribution of the estimates
- Confidence intervals
- The F-test , R-squared

Checking the model
- Fitted values, residuals and leverage
- Extending the model

Why do we need a multiple regression

The simple linear regression model only models how the dependent variable, y, depend on **one** independent variable (covariate) , x_1.

We are often interested in **how** several independent variables, x_1 , x_2 ,..., x_k , influence the dependent variable , y.

Sometimes we want to **adjust** the influence of some of the information, such as age and sex, before we look at the 'effect' of other variables.

A multiple linear regression model

We will here start by considering a **random** subsample consisting of 200 persons from the Frammingham data set used in the book.

A **multiple** linear regression model:

$$\ln(sbp) = \beta_0 + \beta_1 \cdot age + \beta_2 \cdot woman + \beta_3 \cdot \ln(bmi) + E$$

Where the **errors**, E, are assumed to be **independent** and **normal** with mean zero and standard deviation σ.

Note, that the variable $woman$ is a **dummy**/indicator variable, that it is
- **one** if the person is a **woman** and
- **zero** if it is a **man**.

Interpretation of the coefficients 0 – the constant

$$\ln(sbp) = \boxed{\beta_0} + \beta_1 \cdot age + \beta_2 \cdot woman + \beta_3 \cdot \ln(bmi) + E$$

The first coefficient (the constant term) is the **expected** $\ln(sbp)$ for

a **man**	(that is ok!)	
age=0	??????	
bmi=1 kg/m²	??????	($\ln(1)=0$).

As in the simple linear regression this is not of any interest.

But again we can control the interpretation, by choosing **relevant reference** values for age and bmi. E.g.

$$\ln(sbp) = \alpha_0 + \beta_1 \cdot (age - 45) + \beta_2 \cdot woman + \beta_3 \cdot \ln\left(\frac{bmi}{25}\right) + E$$

age45 lnBMI25

Interpretation of the coefficients 1

$$\ln(sbp) = \beta_0 + \boxed{\beta_1} \cdot age + \beta_2 \cdot woman + \beta_3 \cdot \ln(bmi) + E$$

The **expected** $\ln(sbp)$ for a **man** with bmi=27 kg/m² is:

$$\beta_0 + \beta_1 \cdot age + \beta_3 \cdot \ln(27)$$

The **expected** $\ln(sbp)$ for another **man** with the same bmi, but 1.7 **year older**:
$$\beta_0 + \beta_1 \cdot (age + 1.7) + \beta_3 \cdot \ln(27)$$

The difference is: $1.7\beta_1$

We see that this difference

•**does not** depend on the age of the first man.

•**does not** depend on the bmi as long as it is the same for the two men.

•would be the same if the two persons were women.

Interpretation of the coefficients 2

$$\ln(sbp) = \beta_0 + \beta_1 \cdot age + \boxed{\beta_2} \cdot woman + \beta_3 \cdot \ln(bmi) + E$$

The **expected** $\ln(sbp)$ for a 50 year old **man** with bmi=27 kg/m² is:
$$\beta_0 + \beta_1 \cdot 50 \quad + \beta_3 \cdot \ln(27)$$

The **expected** $\ln(sbp)$ for **woman** with the same age and bmi
$$\beta_0 + \beta_1 \cdot 50 + \beta_2 \quad + \beta_3 \cdot \ln(27)$$

The difference is: β_2

We see that this difference

•**does not** depend on the age as long as it is the same for the two persons.

•**does not** depend on the bmi as long as it is the same for the two persons.

Interpretation of the coefficients 3

$$\ln(sbp) = \beta_0 + \beta_1 \cdot age + \beta_2 \cdot woman + \boxed{\beta_3} \ln(bmi) + E$$

The **expected** $\ln(sbp)$ for a **woman** who is 50 year old:

$$\beta_0 + \beta_1 \cdot 50 + \beta_2 + \beta_3 \cdot \ln(bmi)$$

The **expected** $\ln(sbp)$ for another **woman** with the same age, but with a bmi which is 10% higher:

$$\beta_0 + \beta_1 \cdot 50 + \beta_2 + \beta_3 \cdot \ln(1.1 \cdot bmi)$$

The difference $\beta_3 \cdot \left[\ln(1.1 \cdot bmi) - \ln(bmi)\right] = \beta_3 \cdot \ln(1.1)$

We see that this difference

- **does not** depend on the bmi of the first woman.
- **does not** depend on the age as long as it is the same for the two women.
- would be the same if the two persons were **men**.

Interpretation of the coefficients 4

$$\ln(sbp) = \beta_0 + \beta_1 \cdot age + \beta_2 \cdot woman + \boxed{\beta_3} \ln(bmi) + E$$

$$\beta_3 \cdot \left[\ln(1.1 \cdot bmi) - \ln(bmi)\right] = \beta_3 \cdot \ln(1.1)$$

As the bmi is introduced on the **log-scale**, then "differences" of this variable is measured **relatively**.

So comparing a pair of persons who **only differ** in bmi. One having bmi=25 kg/m² and the other bmi=27 kg/m².

Then the expected difference in $\ln(sbp)$ is:

$$\beta_3 \cdot \ln\left(\frac{27}{25}\right) = \beta_3 \cdot 0.077$$

If the bmi's were 21 kg/m² and 23 kg/m², then the expected difference in $\ln(sbp)$ would be:

$$\beta_3 \cdot \ln\left(\frac{23}{21}\right) = \beta_3 \cdot 0.091$$

Interpretation of the coefficients 5

$$\boxed{\ln(sbp)} = \beta_0 + \beta_1 \cdot age + \beta_2 \cdot woman + \beta_3 \cdot \ln(bmi) + E$$

Taking the **exponential** we get:

$$sbp = \gamma_0 \cdot \gamma_1^{age} \cdot \gamma_2^{woman} \cdot bmi^{\beta_3} \cdot \exp(E)$$

where $\gamma_0 = \exp(\beta_0)$, $\gamma_1 = \exp(\beta_1)$ and $\gamma_2 = \exp(\beta_2)$

That is a non-linear model on the sbp scale!

The error is **multiplicative**.

As **medians** are preserved by the exponential transformation then the estimates are measuring the **effects on the median** sbp.

An example: The age and bmi adjusted median sbp is a factor γ_2 higher for men compared to women.

The multiple linear regression in general

Y the **dependent** variable

$(x_1, x_2, \ldots, x_k)$ the **independent** variables.

$$Y = \beta_0 + \sum_{p=1}^{k} \beta_p \cdot x_p + E \qquad E \sim \qquad)$$

This model is based on the **assumptions**:

1. The **expected** value of Y is $\beta_0 + \sum_{p=1}^{k} \beta_p \cdot x_p$
2. The **unexplained** random deviations are **independent**.
3. The unexplained random deviations have the **same distributions**.
4. This distribution is **normal**.

The multiple linear regression in general

$$Y = \beta_0 + \sum_{p=1}^{k} \beta_p \cdot x_p + E \qquad E \sim \qquad)$$

We see that the assumptions fall in **two parts**:

The **first concerning** the systematic part

and the three other which focus on the error, the unexplained random variation.

Before we turn to how one can check some of the assumptions, we will take a closer look at the first assumption.

The **expected** value of Y is $\beta_0 + \sum_{p=1}^{k} \beta_p \cdot x_p$

The assumption of linearity

The **expected** value of Y is $\beta_0 + \sum_{p=1}^{k} \beta_p \cdot x_p$

This is based on three (sub) assumptions:

a. **Additivity**: The contribution from each of the independent variables are **added**.

b. **Proportionality**: The contribution from a independent variable is **proportional** to its value (with a factor β)

c. **No effectmodification**: The contribution from one independent variables **is the same** whatever the values are for the other.

The assumption of linearity

The **expected** value of Y is $\beta_0 + \sum_{p=1}^{k} \beta_p \cdot x_p$

If one consider two persons who differ with

Δx_1 in x_1 , Δx_2 in x_2 ... and Δx_k in x_k

then the difference in the **expected** value of Y is :

$$\sum_{p=1}^{k} \beta_p \cdot \Delta x_p$$

Again we see that the **contribution** for each of the explanatory variables:

are **added**,

are **proportional** to the difference

and **does not dependent** of the differences in the other

Estimation

It is almost impossible to find the estimates by hand, but easy if you use a computer.

In Stata: ***regress*** **lnSBP age45 woman lnBMI25**

(Note first we have to generate `lnSBP, age45, woman and lnBMI25`)

Source	SS	df	MS
Model	1.05572698	3	.351908994
Residual	4.18969066	196	.021375973
Total	5.24541764	199	.026358883

Number of obs	= 200
F(3, 196)	= 16.46
Prob > F	= 0.0000
R-squared	= 0.2013
Adj R-squared	= 0.1890
Root MSE	= .14621

lnSBP	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
woman	.0036329	.0208905	0.17	0.862	-.0375662	.0448319
age45	.0065384	.0012844	5.09	0.000	.0040053	.0090715
lnBMI25	.2583399	.0758295	3.41	0.001	.1087934	.4078864
_cons	4.856592	.0154266	314.82	0.000	4.826169	4.887016

Estimation

The last part of the output: No CI for σ ! It should be calculated "by hand"

$\hat{\sigma}$

Root MSE = .14621

lnSBP	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
woman	.0036329	.0208905	0.17	0.862	-.0375662	.0448319
age45	.0065384	.0012844	5.09	0.000	.0040053	.0090715
lnBMI25	.2583399	.0758295	3.41	0.001	.1087934	.4078864
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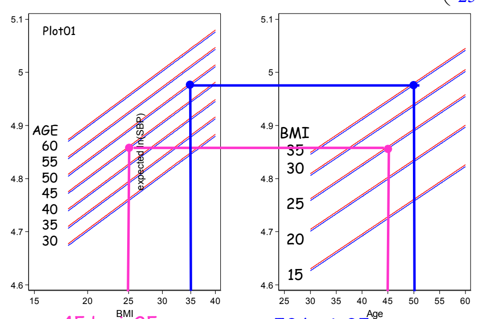
the $\hat{\beta}$'s the se's The CI 's

Test for $\beta_2 = 0$

The hypothesis: "no difference in $\ln(sbp)$ between men and women **adjusted** for age and bmi"

Estimated systematic part

$$\ln(sbp) = 4.8566 + 0.0065 \cdot (age - 45) + 0.0036 \cdot woman + 0.2583 \cdot \ln\left(\frac{bmi}{25}\right)$$



age=45 bmi=25 age=50 bmi=35

Stata special – plotting response curves

regress lnSBP age45 woman lnBMI25

lnSBP	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
woman	.0036329	.0208905	0.17	0.862	-.0375662	.0448319
age45	.0065384	.0012844	5.09	0.000	.0040053	.0090715
lnBMI25	.2583399	.0758295	3.41	0.001	.1087934	.4078864
_cons	4.856592	.0154266	314.82	0.000	4.826169	4.887016

After a regression commando, Stata leaves several information in the memory of the computer for later use.

You can get a list by writing "`ereturn list` ".
We have already used this feature in the calculation of the confidence interval for σ.

Another example:

```
. display %12.7f _b[woman] %12.7f _se[woman]
0.0036329   0.0208905
```


Stata special – plotting "response" curves

I have made a Stata command that extracts the estimated equations and the coefficients for later use.
The command file
`regeq.ado`
and the small help file
`regeq.sthlp`
should be place in your ado folder typically
`c:\ado\personal.`

You can run the `regeq` command after any linear or logistic regression estimation.
Here you get the output :

```
estimated equation
4.85659  +0.003632 * woman  +0.006538 * age45 +0.25834* lnBMI25
equation
b0 + b1 * woman + b2 * age45 + b3 * lnBMI25
```

That is, the estimated equation and the formula.

Stata special – plotting "response" curves

Furthermore the estimated coefficients are stored as " global macros":

```
. macro list
b0:             4.856592269392944
b3:             .2583398993331004
b2:             .0065383788673611
b1:             .0036328605876014

S_E_depv:       lnSBP
S_E_cmd:        regress
.....
```

The global macros b0 to b3 contains the coefficients and can be used in calculations.
If you want to use the estimated coefficient to age45, then you just write $b2.

Stata special – plotting "response" curves

The expected log(SBP) for a 30 year old man with BMI=27
remember: `Y= b0 + b1 * woman + b2 * age45 + b3 * lnBMI25`

```
display $b0+$b1*0+$b2*(30-45) +$b3*ln(27/25)
4.7783987
```

You could also get this (with CI) using the lincom command:

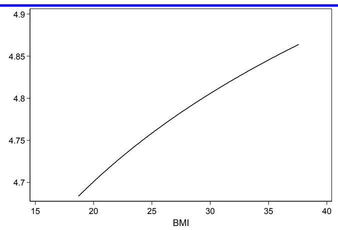
```
 display ln(27/25)
.07696104

. lincom -15*age45 + .07696104*lnBMI25+_cons

 ( 1) - 15 age45 + .076961 lnBMI25 + _cons = 0

------------------------------------------------------------------------------
       lnSBP |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         (1) |   4.778399   .0266891   179.04   0.000     4.725764    4.831033
------------------------------------------------------------------------------
```


Remember: `Y = b0+ b1 * woman + b2 * age45 + b3 * lnBMI25`
The expected log(SBP) for a 30 year old man as a function of the **BMI** is given as:

`Y = b0 + b1 *0 +b2 * (30-45) + b3 * ln(BMI/25)`

We can plot this by using the plot function in Stata:

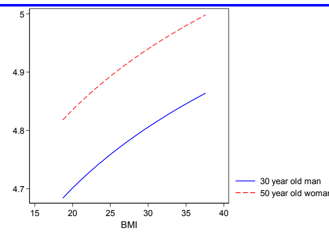
```
 twoway                                                                    ///
( function Y=$b0 + $b1 * 0 +$b2 * (30-45) + $b3 * ln(x/25), range(bmi) ) ///
, legend(off) ytit("expected ln(SBP)") xtit("BMI") xlab( 15(5)40)
```



Stata special – plotting response curves

The expected log(SBP) for a 30 year old man and a 50 year old woman as a function of the BMI is given as:

```
 twoway                                                              ///
( function Y=$b0 + $b1 * 0 + $b2 * (30-45) + $b3 * ln(x/25)          ///
                                        , range(bmi) lco(blue) ) ///
( function Y=$b0 + $b1 * 1 + $b2 * (50-45) + $b3 * ln(x/25)          ///
                                        , range(bmi) lco(red)  ) ///
,  ytit("expected ln(SBP)") xtit("BMI") xlab( 15(5)40)               ///
 legend(label(1 "30 year old man") label(2 "50 year old woman"))
```



The distribution of the estimates

It can be shown that the **estimates of the coefficients** have **normal** distributions, with **means** equal to the **true values**.

The formulas for the standard deviation of the estimates are **complicated**, but they are estimated by the **standard errors** given in the output.

The estimated standard deviation of the errors is given by:

$$\hat{\sigma}^2 \sim \frac{\sigma^2}{n-k-1}\cdot\chi^2\left(n-k-1\right)$$

The number of parameters are $k+1$

Which gives the confidence interval:

$$95\%\text{ CI for }\sigma : \hat{\sigma}\cdot\sqrt{\frac{n-k-1}{\chi^2_{n-k-1}(0.975)}} \le \sigma \le \hat{\sigma}\cdot\sqrt{\frac{n-k-1}{\chi^2_{n-k-1}(0.025)}}$$

You can use the Stata command `cisd`

Confidence intervals

Just like in the simple regression we get :
(except we have n-k-1 degrees of freedom).

Exact 95% confidence intervals , CI's, for β_p is found from the estimates and standard errors

$$95\%\text{ CI for }\beta_p : \hat{\beta}_p \pm t^{0.975}_{n-k-1}\cdot\mathrm{se}\left(\hat{\beta}_p\right)$$

Where $t^{0.975}_{n-k-1}$ is the upper 97.5 percentile in the t-distribution n-k-1 degrees of freedom.

These confidence intervals are found in the output.

Note that if n-k-1 is large then this percentile is close to 1.96 and one can use the **approximate confidence intervals:**

$$\text{Approx. }95\%\text{ CI for }\beta_p : \hat{\beta}_p \pm 1.96\cdot\mathrm{se}\left(\hat{\beta}_p\right)$$

The ANOVA table and the F-test

The first part of the output:

An **an**alysis **o**f **va**riance table dividing the variation in y in two components: explained by the **model** (i.e. the **3** variables) and the **residual** (the rest)

```
   Source |       SS       df       MS
----------+------------------------------
    Model |  1.05572698     3  .351908994
 Residual |  4.18969066   196  .021375973
----------+------------------------------
    Total |  5.24541764   199  .026358883
```

```
Number of obs =     200
F(  3,   196) =   16.46
Prob > F      =  0.0000
R-squared     =  0.2013
Adj R-squared =  0.1890
Root MSE      =  .14621
```

A F-test testing the hypothesis: "all (except β_0) is zero."

Here the test is highly significant: The model explains a statistically significant part of the variation in y!

The F-test and R-squared

The F- test calculated as: $F = \dfrac{0.35519}{0.02138} = 16.46$

```
   Source |       SS       df       MS          Number of obs =     200
----------+------------------------------      F(  3,   196) =   16.46
    Model |  1.05572698     3  .351908994      Prob > F      =  0.0000
 Residual |  4.18969066   196  .021375973      R-squared     =  0.2013
----------+------------------------------      Adj R-squared =  0.1890
    Total |  5.24541764   199  .026358883      Root MSE      =  .14621
```

And under the hypothesis it follows an F-distribution with 3 and 196 degrees of freedom.

The `R-squared` is the amount of the total variation explained by the model(=$1.0557/5.2454$).

As this will **increase**, if we include more variables in the model, one can look at the **`adjusted R-squared`**.

Predicted values, residuals and leverages

$$Y = \beta_0 + \sum_{p=1}^{k} \beta_p \cdot x_p + E \quad E \sim \qquad)$$

As in the simple linear regression one can find **predicted** values, **residuals, leverages** and **standardized residuals**:

Predicted value: $\hat{y}_i = \hat{\beta}_0 + \sum_{p=1}^{k} \hat{\beta}_p \cdot x_{pi}$

Residual: $r_i = y_i - \hat{y}_i = y_i - \left(\hat{\beta}_0 + \sum_{p=1}^{k} \hat{\beta}_p \cdot x_{pi} \right)$

Leverage: $h_i =$ a complicated formula

Standardized-Residual: $z_i = \dfrac{r_i}{\hat{\sigma}\sqrt{1-h_i}}$

Leverage

Although the formula for the leverage is complicated, the **interpretation** of leverage is the same:

A **high leverage** indicates that the data point has **extreme** values of the explanatory variables and hence a **high influence** on the estimates.

Checking the model 1:

As the model is much more complicated than the simple linear regression checking the model is also complicated

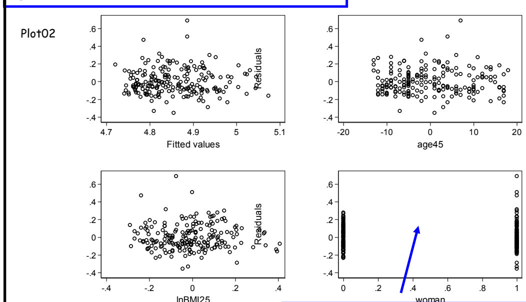
Again **assumption no. 2**: *the errors should be independent*, is mainly checked by considering how the data was collected.

The **distribution of the error** is checked by the same type of plot as for the simple linear regression.

- Plots of residuals versus **fitted**
- Plots of residuals versus **each of** the **explanatory** variables.
- Histogram and QQ-plot of the residuals.


```
rvfplot            ,name(p1,replace)
rvpplot age45      ,name(p2,replace)
rvpplot lnBMI25    ,name(p3,replace)
rvpplot woman      ,name(p4,replace)
graph combine p1 p2 p3 p4
```

residual **v**ersus **f**itted
residual **v**ersus **p**redictor



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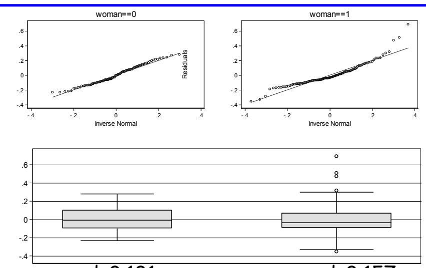
Diagnostic plots for categorical variables – here woman

```
predict res if e(sample),res
qnorm res if woman==0, title(woman==0) name(p1,replace)
qnorm res if woman==1, title(woman==1) name(p2,replace)
graph combine p1 p2  , row(1)          name(p3,replace)
graph box res        , over(woman)     name(p4,replace)
graph combine p3 p4,col(1)
by woman: sum res
```

???

Plot03

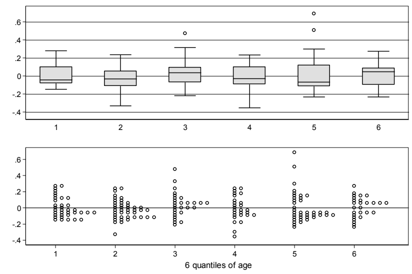


sd=0.131 sd=0.157

Diagnostic plots for continuous variables – dividing into groups

```
xtile age6=age,nq(6)
graph box res,over(age6)             name(p1,replace) nodraw
dotplot   res,over(age6) yline(0)   name(p2,replace) nodraw
graph combine p1 p2 ,col(1)
graph export Reg2_1_plot04.wmf, replace
```

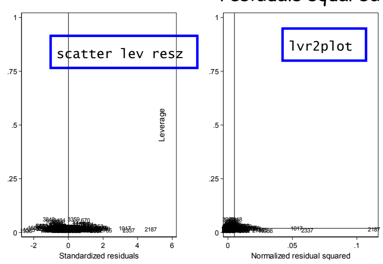
Plot04



Identifying special points

leverage vs. residuals leverage vs. normed residuals squared#

Plot05



1017, 2337, 2187 have relative large residuals

#: $r_i^2 \Big/ \sum r_j^2$

Checking the model 2: Independent errors ?

Assumption no. 2: *the errors should be **independent***, is mainly checked by considering **how the data was collected**.

The assumption is **violated** if

•some of the persons are **relatives** (and some are not) and the dependent variable have some **genetic** component.

•some of the persons were **measured** using one instrument and others with another.

•in general if the persons were sampled in clusters.

Checking the model 3: Extending the model

One should **also** try to check the validity of the linearity assumption that is the assumption of **additivity**, **proportionality** and **no effect modification** (no interaction).

It can be done by:

1. Introducing the explanatory variable in a **different scale**, e.g. adding age^2 or $\log(age)$ ….
2. Introducing the explanatory variable as a **categorical** variable instead e.g. use *age* divided into **agegroups** instead as age in years.
3. Introducing **interactions** between some of the eplanatory variables.
4. ….